Review Session:

1. Functions of Continuous Random Variable

Theorem: Let X be continuous random variable with density function $f_X(x)$. Suppose $g(x)$ is increasing or decreasing continuous function. $Y=g(X)$ has pdf given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| & \text{if } y=g(x) \text{ for some } x. \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

Example 1: Let X be a continuous random variable with PDF $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, for $x \in \mathbb{R}$, and let $Y = X^2$. Find $f_Y(y)$.

$g(x) = x^2$, then $g^{-1}(y) = \pm\sqrt{y}$. $\frac{d}{dy} g^{-1}(y) = \pm \frac{1}{2\sqrt{y}}$.

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|} = \sum_{i=1}^{n} f_X(x_i) \cdot \left|\frac{dx_i}{dy}\right|.$$

$$f_Y(y) = \frac{f_X(\sqrt{y})}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})}{|-2\sqrt{y}|} = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}, \text{ for } y \in (0,\infty)$$

Example 2: Let X be continuous r.v. with PDF

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \le 1. \\ 0 & \text{otherwise} \end{cases}$$ and let $Y = \frac{1}{X}$. Find $f_Y(y)$.

$g(x) = \frac{1}{x}$ is decreasing on $(0,1]$, and $g^{-1}(y) = \frac{1}{y}$. and $y \geq 1$.

$\left|\frac{d}{dy} g^{-1}(y)\right| = \left|-\frac{1}{y^2}\right| = \frac{1}{y^2}$. $\quad f(y) = \frac{4}{y^3} \cdot \frac{1}{y^2} = \frac{4}{y^5}$.

$$f_Y(y) = \begin{cases} \frac{4}{y^5} & y \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. Bivariate transformation.

Theorem: $X = (X_1, X_2)$ with joint PDF $f(x, y)$, consider new random vector $Y = (Y_1, Y_2)$.

$Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$. To get the joint PDF of Y, we need to

① solve $X_1 = h_1(Y_1, Y_2)$, $X_2 = h_2(Y_1, Y_2)$.

② find Jacobian $J = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{vmatrix}$.

③ the joint PDF is $f(h_1(y_1, y_2), h_2(y_1, y_2))|J|$.

Example ① ~~(X, Y)~~ X, Y are independent standard normal random variable.

With the transformation $U = X+Y$ and $V = X-Y$. ~~In the notation~~

Find Joint PDF of U and V. Whether they are independent.

① $X = \frac{U+V}{2}$, $Y = \frac{U-V}{2}$.

② $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$, $|J| = \frac{1}{2}$.

③ $f_{U,V}(u, v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot \frac{1}{2} = \frac{1}{2\pi} e^{-\left(\frac{u+v}{2}\right)^2/2 - \left(\frac{u-v}{2}\right)^2/2}$

$= \left(\frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} \cdot e^{-\frac{u^2}{4}}\right) \cdot \left(\frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} e^{-\frac{v^2}{2}}\right)$.

④ They are independent.

3. Conditional distribution.

If X_1 and X_2 are not independent, $f_{X_1}(x \mid X_2 = y) = \dfrac{f(x,y)}{f_{X_2}(y)}$.

If X_1 and X_2 are ~~independent~~ independent. $f_{X_1}(x \mid X_2 = y) = f_{X_1}(x)$.

Example: The joint PDF of X and Y is given by

$$f(x,y) = \begin{cases} \frac{12}{5} x(2-x-y), & 0<x<1,\ 0<y<1. \\ 0 & \text{otherwise}. \end{cases}$$

Compute the conditional PDF of X given $Y=y$.

$$f(x \mid y) = \frac{f(x,y)}{f(y)} = \frac{\frac{12}{5}x(2-x-y)}{\int_0^1 \frac{12}{5}x(2-x-y)\,dx} = \frac{\frac{12}{5}x(2-x-y)}{\frac{12}{5}\int_0^1 2x - x^2 - xy\,dx}$$

$$= \frac{x(2-x-y)}{x^2 - \frac{x^3}{3} - \frac{x^2 y}{2}\Big|_0^1} = \frac{x(2-x-y)}{\frac{2}{3} - \frac{y}{2}}. \qquad 0<y<1.$$

$$f(x \mid y) = \begin{cases} \frac{6x(2-x-y)}{4-3y}, & 0<x<1,\ 0<y<1. \\ 0, & \text{otherwise}. \end{cases}$$

4. Covariance: The covariance between X and Y is

$$\operatorname{cov}(X,Y) = E[(X - EX)(Y - EY)] = EXY - EXEY.$$

~~Example: Let the joint pdf of (X,Y) be $f(x,y) = 1$, $0<x<1$, $x<y<x+1$~~
~~The marginal distribution of X is uniform $(0,1)$. Find the covariance of X and Y.~~
~~Marginal distribution of Y. $\int_0^1 f(x,y)\,dx =$~~

Example:

Let Z_1 and Z_2 be independent $N(0,1)$ random variables. define new random variable X and Y by

$$X = a_X Z_1 + b_X Z_2 + C_X, \quad Y = a_Y Z_1 + b_Y Z_2 + C_Y.$$

Show that
$$EX = C_X, \quad Var X = a_X^2 + b_X^2$$
$$EY = C_Y, \quad Var Y = a_Y^2 + b_Y^2$$
$$Cov(X,Y) = a_X a_Y + b_X b_Y.$$

5. Moment Generating function.

$$M(t) = Ee^{tX}, \quad EX = M'(0), \quad EX^2 = M''(0), \ \ldots \ EX^n = M^{(n)}(0).$$

Example: If $X \sim Poisson(\lambda)$, $Y \sim Poisson(\mu)$. they are independent prove $X+Y \sim Poisson(\lambda+\mu)$.

By the proposition, X and Y are independent if and only if $M_X(t)\, M_Y(s) = M_{X,Y}(t,s)$.

MGF for X is
$$M_X(t) = Ee^{tX} = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{(e^t\lambda)^n}{n!}$$
$$= e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t-1)}.$$

Similarly $M_Y(t) = e^{\mu(e^t-1)}$.

By the independence $M_X(t) \cdot M_Y(t) = e^{(\mu+\lambda)(e^t-1)}$.

while $M_X(t) \cdot M_Y(t) = Ee^{tX} \cdot Ee^{tY} = Ee^{t(X+Y)} = M_{X+Y}(t)$.

then $X+Y \sim Poisson(\lambda+\mu)$.